On U-Dominant Dimension[*†]

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Abstract

Let Λ and Γ be artin algebras and ${}_\Lambda U_\Gamma$ a faithfully balanced selforthogonal bimodule. We show that the U-dominant dimensions of ${}_\Lambda U$ and U_Γ are identical. As applications to the results obtained, we give some characterizations of double dual functors (with respect to ${}_\Lambda U_\Gamma$) preserving monomorphisms and being left exact respectively.

1. Introduction

For a ring Λ, we use mod Λ (resp. mod Λ^{op}) to denote the category of finitely generated left Λ-modules (resp. right Λ-modules).

Definition 1.1 Let Λ and Γ be rings. A bimodule ${}_\Lambda T_\Gamma$ is called a faithfully balanced selforthogonal bimodule if it satisfies the following conditions:

(1) ${}_\Lambda T \in$mod Λ and $T_\Gamma \in$mod Γ^{op}.

(2) The natural maps $\Lambda \to \mathrm{End}(T_\Gamma)$ and $\Gamma \to \mathrm{End}({}_\Lambda T)^{op}$ are isomorphisms.

(3) $\mathrm{Ext}^i_\Lambda({}_\Lambda T, {}_\Lambda T) = 0$ and $\mathrm{Ext}^i_\Gamma(T_\Gamma, T_\Gamma) = 0$ for any $i \geq 1$.

Definition 1.2 Let U be in mod Λ (resp. mod Γ^{op}) and n a non-negative integer. For a module M in mod Λ (resp. mod Γ^{op}),

(1)[8] M is said to have U-dominant dimension greater than or equal to n, written U-dom.dim$({}_\Lambda M)$ (resp. U-dom.dim(M_Γ)) $\geq n$, if each of the first n terms in a minimal injective resolution of M is cogenerated by ${}_\Lambda U$ (resp. U_Γ), that is, each of these terms can be embedded into a direct product of copies of ${}_\Lambda U$ (resp. U_Γ).

(2)[10] M is said to have dominant dimension greater than or equal to n, written dom.dim$({}_\Lambda M)$ (resp. dom.dim(M_Γ)) $\geq n$, if each of the first n terms in a minimal injective resolution of M is Λ-projective (resp. Γ^{op}-projective).

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Assume that Λ is an artin algebra. By [4] Theorem 3.3, Λ^I and each of its direct summands are projective for any index set I. So, when ${}_\Lambda U = {}_\Lambda \Lambda$ (resp. $U_\Gamma = \Lambda_\Lambda$), the notion of U-dominant dimension coincides with that of (classical) dominant dimension. Tachikawa in [10] showed that if Λ is a left and right artinian ring then the dominant dimensions of ${}_\Lambda\Lambda$ and Λ_Λ are identical. Kato in [8] characterized the modules with U-dominant dimension greater than or equal to one. Colby and Fuller in [5] gave some equivalent conditions of dom.dim$({}_\Lambda\Lambda) \geq 1$ (or 2) in terms of the properties of double dual functors (with respect to ${}_\Lambda\Lambda_\Lambda$).

The results mentioned above motivate our interests in establishing the identity of U-dominant dimensions of ${}_\Lambda U$ and U_Γ and characterizing the properties of modules with a given U-dominant dimension. Our characterizations will lead a better comprehension about U-dominant dimension and the theory of selforthogonal bimodules.

Throughout this paper, Λ and Γ are artin algebras and ${}_\Lambda U_\Gamma$ is a faithfully balanced selforthogonal bimodule. The main result in this paper is the following

Theorem 1.3 U-dom.dim$({}_\Lambda U) = U$-dom.dim(U_Γ).

Put ${}_\Lambda U_\Gamma = {}_\Lambda\Lambda_\Lambda$, we immediately get the following result, which is due to Tachikawa (see [10]).

Corollary 1.4 dom.dim$({}_\Lambda\Lambda)$ =dom.dim(Λ_Λ).

Let M be in mod Λ (resp. mod Γ^{op}) and $G(M)$ the subcategory of mod Λ (resp. mod Γ^{op}) consisting of all submodules of the modules generated by M. M is called a QF-3 module if $G(M)$ has a cogenerator which is a direct summand of every other cogenerator[11]. By [11] Proposition 2.2 we have that a finitely cogenerated Λ-module (resp. Γ^{op}-module) M is a QF-3 module if and only if M cogenerates its injective envelope. So by Theorem 1.3 we have

Corollary 1.5 ${}_\Lambda U$ *is* QF-3 *if and only if* U_Γ *is* QF-3.

We shall prove our main result in Section 2. As applications to the results obtained in Section 2, we give in Section 3 some characterizations of double dual functors (with respect to ${}_\Lambda U_\Gamma$) preserving monomorphisms and being left exact respectively.

2. The proof of main result

Let E_0 be the injective envelope of ${}_\Lambda U$. Then E_0 defines a torsion theory in mod Λ. The torsion class $\mathcal{T}$ is the subcategory of mod Λ consisting of the modules X satisfying $\mathrm{Hom}_\Lambda(X, E_0) = 0$, and the torsionfree class $\mathcal{F}$ is the subcategory of mod Λ consisting of the

modules Y cogenerated by E_0 (equivalently, Y can be embedded in E_0^I for some index set I). A module in mod Λ is called torsion (resp. torsionfree) if it is in $\mathcal{T}$ (resp. $\mathcal{F}$). The injective envelope $E_0^{'}$ of U_Γ also defines a torsion theory in mod Γ^{op} and we may give in mod Γ^{op} the corresponding notions as above. Let X be in mod Λ (resp. mod Γ^{op}) and $t(X)$ the torsion submodule, that is, $t(X)$ is the submodule X such that $\mathrm{Hom}_\Lambda(t(X), E_0) = 0$ (resp. $\mathrm{Hom}_\Gamma(t(X), E_0^{'}) = 0$) and E_0 (resp. $E_0^{'}$) cogenerates $X/t(X)$ (c.f. [7]).

Let A be in mod Λ (resp. mod Γ^{op}). We call $\mathrm{Hom}_\Lambda({}_\Lambda A, {}_\Lambda U_\Gamma)$ (resp. $\mathrm{Hom}_\Gamma(A_\Gamma, {}_\Lambda U_\Gamma)$) the dual module of A with respect to ${}_\Lambda U_\Gamma$, and denote either of these modules by A^*. For a homomorphism f between Λ-modules (resp. Γ^{op}-modules), we put $f^* = \mathrm{Hom}(f, {}_\Lambda U_\Gamma)$. Let $\sigma_A : A \to A^{**}$ via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^*$ be the canonical evaluation homomorphism. A is called U-torsionless (resp. U-reflexive) if σ_A is a monomorphism (resp. an isomorphism).

Lemma 2.1 *For a module X in* mod Λ (*resp.* mod Γ^{op}), $t(X) = \mathrm{Ker}\sigma_X$ *if and only if* $\mathrm{Hom}_\Lambda(\mathrm{Ker}\sigma_X, E_0) = 0$ (*resp.* $\mathrm{Hom}_\Gamma(\mathrm{Ker}\sigma_X, E_0^{'}) = 0$).

Proof. The necessity is trivial. Now we prove the sufficiency.

We have the following commutative diagram with the upper row exact:

$$\begin{tikzcd}
0 \arrow[r] & t(X) \arrow[r] & X \arrow[r, "\pi"] \arrow[d, "\sigma_X"] & X/t(X) \arrow[r] \arrow[d, "\sigma_{X/t(X)}"] & 0 \\
& & X^{**} \arrow[r, "\pi^{**}"] & {[X/t(X)]^{**}} &
\end{tikzcd}$$

Since $\mathrm{Hom}_\Lambda(t(X), E_0) = 0$, $[t(X)]^* = 0$ and π^* is an isomorphism. So π^{**} is also an isomorphism and hence $t(X) \subset \mathrm{Ker}\sigma_X$. On the other hand, $\mathrm{Hom}_\Lambda(\mathrm{Ker}\sigma_X, E_0) = 0$ by assumption, which implies that $\mathrm{Ker}\sigma_X$ is a torsion module and contained in X. So we conclude that $\mathrm{Ker}\sigma_X \subset t(X)$ and $\mathrm{Ker}\sigma_X = t(X)$. ■

Remark. From the above proof we always have $t(X) \subset \mathrm{Ker}\sigma_X$.

Suppose that $A \in$ mod Λ (resp. mod Γ^{op}) and $P_1 \xrightarrow{f} P_0 \to A \to 0$ is a (minimal) projective resolution of A. Then we have an exact sequence $0 \to A^* \to P_0^* \xrightarrow{f^*} P_1^* \to \mathrm{Coker} f^* \to 0$. We call $\mathrm{Coker} f^*$ the transpose (with respect to ${}_\Lambda U_\Gamma$) of A, and denote it by $\mathrm{Tr}_U A$.

Proposition 2.2 *The following statements are equivalent.*

(1) $t(X) = \mathrm{Ker}\sigma_X$ *for every* $X \in$ mod Λ.

(2) f^{**} *is monic for every monomorphism* $f : A \to B$ *in* mod Λ.

$(1)^{op}$ *$t(Y)=$*Kerσ_Y *for every* $Y\in$mod Γ^{op}.

$(2)^{op}$ *g^{**} is monic for every monomorphism $g: C\to D$ in* mod Γ^{op}.

Proof. By symmetry, it suffices to prove the implications of $(1)\Rightarrow(2)^{op}\Rightarrow(1)^{op}$.

$(1)\Rightarrow(2)^{op}$ Let $g: C\to D$ be monic in mod Γ^{op}. Set $X=$Coker$g$. We have that Ker$\sigma_{\mathrm{Tr}_U X}\cong$Ext$^1_\Lambda(X,U)$ and Tr$_U X\in$mod Λ by [6] Lemma 2.1. By (1) and Lemma 2.1, Hom$_\Gamma$(Ext$^1_\Lambda(X,U),E_0)=0$. Since Cokerg^* can be imbedded in Ext$^1_\Lambda(X,U)$, Hom$_\Gamma$(Cokerg^*, $E_0)=0$. But (Coker$g^*)^*\subset$ Hom$_\Gamma$(Cokerg^*,E_0), so (Coker$g^*)^*=0$ and hence Ker$g^{**}\cong$ (Coker$g^*)^*=0$, which implies that g^{**} is monic.

$(2)^{op}\Rightarrow(1)^{op}$ Let Y be in mod Γ^{op} and X any submodule of Kerσ_Y and $f_1: X\to$Kerσ_Y the inclusion. Assume that f is the composition: $X\xrightarrow{f_1}$Ker$\sigma_Y\to Y$. Then $\sigma_Y f=0$ and $f^*\sigma_Y^*=(\sigma_Y f)^*=0$. But σ_Y^* is epic by [1] Proposition 20.14, so $f^*=0$ and $f^{**}=0$. By $(2)^{op}$, f^{**} is monic, so $X^{**}=0$ and $X^{***}=0$. Since X^* is isomorphic to a submodule of X^{***} by [1] Proposition 20.14, $X^*=0$.

We claim that Hom$_\Gamma$(Ker$\sigma_Y,E_0^{'})=0$. Otherwise, there exists $0\neq\alpha\in$ Hom$_\Gamma$(Ker$\sigma_Y,E_0^{'}$). Then Im$\alpha\bigcap U_\Gamma\neq0$ since U_Γ is an essential submodule of $E_0^{'}$. So α^{-1}(Im$\alpha\bigcap U_\Gamma$) is a non-zero submodule of Kerσ_Y and there exists a non-zero map α^{-1}(Im$\alpha\bigcap U_\Gamma)\to U_\Gamma$, which implies that $(\alpha^{-1}$(Im$\alpha\bigcap U_\Gamma))^*\neq0$, a contradiction with the former argument. Hence we conclude that $t(Y)=$Kerσ_Y by Lemma 2.1. ■

Let A be a Λ-module (resp. a Γ^{op}-module). We denote either of Hom$_\Lambda({}_\Lambda U_\Gamma,{}_\Lambda A)$ and Hom$_\Gamma({}_\Lambda U_\Gamma,A_\Gamma)$ by ${}^*A$, and the left (resp. right) flat dimension of A by $l.$fd$_\Lambda(A)$ (resp. $r.$fd$_\Gamma(A)$). We give a remark as follows. For an artin algebra R and a left (resp. right) R-module A, we have that the left (resp. right) flat dimension of A and its left (resp. right) projective dimension are identical; especially, A is left (resp. right) flat if and only if it is left (resp. right) projective.

Lemma 2.3 *Let ${}_\Lambda E$ (resp. E_Γ) be injective and n a non-negative integer. Then $l.$*fd$_\Gamma({}^*E)$ (*resp.* $r.$fd$_\Lambda({}^*E)$) $\leq n$ *if and only if* Hom$_\Lambda$(Ext$^{n+1}_\Gamma(A,U),E$) (*resp.* Hom$_\Gamma$(Ext$^{n+1}_\Lambda(A,U),E$)) $=0$ *for any $A\in$*mod Γ^{op} (*resp.* mod Λ).

Proof. It is trivial by [3] Chapter VI, Proposition 5.3. ■

Proposition 2.4 *The following statements are equivalent.*

(1) *${}^*E_0$ is flat.*

(2) *There is an injective Λ-module E such that ${}^*E$ is flat and E cogenerates E_0.*

(3) *$t(X)=$*Kerσ_X *for any* $X\in$mod Λ.

Proof. (1) $\Rightarrow$ (2) It is trivial.

(2) $\Rightarrow$ (3) Let $X \in$mod Λ. Since Ker$\sigma_X \cong$Ext$^1_\Gamma$(Tr$_U X, U$) with Tr$_U X \in$mod Γ^{op} by [6] Lemma 2.1. By (2) and Lemma 2.3, Hom$_\Lambda$(Ext$^1_\Gamma$(Tr$_U X, U$)$, E) = 0$.

Since E cogenerates E_0, there is an exact sequence $0 \to E_0 \to E^I$ for some index set I. So Hom$_\Lambda$(Ext$^1_\Gamma$(Tr$_U X, U$)$, E_0$) $\subset$ Hom$_\Lambda$(Ext$^1_\Gamma$(Tr$_U X, U$)$, E^I$) $\cong$ [Hom$_\Lambda$(Ext$^1_\Gamma$(Tr$_U X, U$)$, E$)]^I $=$ 0 and Hom$_\Lambda$(Ext$^1_\Gamma$(Tr$_U X, U$)$, E_0) = 0$. By Lemma 2.1, $t(X) =$Kerσ_X.

(3) $\Rightarrow$ (1) Let $N \in$mod Γ^{op}. Since Ker$\sigma_{\mathrm{Tr}_U N} \cong$Ext$^1_\Gamma(N, U)$ with Tr$_U N \in$mod Λ by [6] Lemma 2.1, By (3) and Lemma 2.1 we have Hom$_\Lambda$(Ext$^1_\Gamma(N, U), E_0) \cong$ Hom$_\Lambda$(Ker$\sigma_{\mathrm{Tr}_U N}, E_0$) $= 0$, and so $^*E_0$ is flat by Lemma 2.3. ■

Dually, we have the following

Proposition 2.4$'$ *The following statements are equivalent.*

(1) $^*E'_0$ *is flat.*

(2) *There is an injective* Γ^{op}*-module* E' *such that* $^*E'$ *is flat and* E' *cogenerates* E'_0.

(3) $t(Y) =$Kerσ_Y *for any* $Y \in$mod Γ^{op}.

Corollary 2.5 $^*E_0$ *is flat if and only if* $^*E_0^{'}$ *is flat.*

Proof. By Propositions 2.2, 2.4 and 2.4$'$. ■

Let $A \in$ mod Λ (resp. mod Γ^{op}) and i a non-negative integer. We say that the grade of A with respect to $_\Lambda U_\Gamma$, written grade$_U A$, is greater than or equal to i if Ext$^j_\Lambda(A, U) = 0$ (resp. Ext$^j_\Gamma(A, U) = 0$) for any $0 \leq j < i$.

Lemma 2.6 *Let* X be in mod Γ^{op} *and* n *a non-negative integer. If* grade$_U X \geq n$ *and* grade$_U$Ext$^n_\Gamma(X, U) \geq n + 1$, *then* Ext$^n_\Gamma(X, U) = 0$.

Proof. Since X^* is U-torsionless, $X^{**} = 0$ if and only if $X^* = 0$. Then the case $n = 0$ follows.

Now let $n \geq 1$ and

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to X \to 0$$

be a projective resolution of X in mod Γ^{op}. Put $X_n =$Coker$(P_{n+1} \to P_n)$. Then we have an exact sequence

$$0 \to P_0^* \to \cdots \to P_{n-1}^* \xrightarrow{f} X_n^* \to \mathrm{Ext}^n_\Gamma(X, U) \to 0$$

in mod Λ with each $P_i^* \in$add$_\Lambda U$. Since grade$_U$Ext$^n_\Gamma(X, U) \geq n+1$, Ext$^i_\Lambda$(Ext$^n_\Gamma(X, U), U) = 0$ for any $0 \leq i \leq n$. So Ext$^i_\Lambda$(Ext$^n_\Gamma(X, U), P_j^*) = 0$ for any $0 \leq i \leq n$ and $0 \leq j \leq n-1$, and hence Ext$^1_\Lambda$(Ext$^n_\Gamma(X, U)$, Imf) $\cong$ Ext$^n_\Lambda$(Ext$^n_\Gamma(X, U), P_0^*) = 0$, which implies that we have an exact sequence Hom$_\Lambda$(Ext$^n_\Gamma(X, U), X_n^*$) $\to$ Hom$_\Lambda$(Ext$^n_\Gamma(X, U)$, Ext$^n_\Gamma(X, U)$) $\to 0$. Notice

that X_n^* is U-torsionless and $\mathrm{Hom}_\Lambda(\mathrm{Ext}_\Gamma^n(X,U),U)=0$. So $\mathrm{Hom}_\Lambda(\mathrm{Ext}_\Gamma^n(X,U),X_n^*)=0$ and $\mathrm{Hom}_\Lambda(\mathrm{Ext}_\Gamma^n(X,U),\ \mathrm{Ext}_\Gamma^n(X,U))=0$, which implies that $\mathrm{Ext}_\Gamma^n(X,U)=0$. ■

Remark. We point out that all of the above results (from 2.1 to 2.6) in this section also hold in the case Λ and Γ are left and right noetherian rings.

For a module T in mod Λ (resp. mod Γ^{op}), we use $\mathrm{add}_\Lambda T$ (resp. $\mathrm{add}T_\Gamma$) to denote the subcategory of mod Λ (resp. mod Γ^{op}) consisting of all modules isomorphic to direct summands of finite direct sums of copies of ${}_\Lambda T$ (resp. T_Γ). Let A be in mod Λ. If there is an exact sequence $\cdots\to U_n\to\cdots\to U_1\to U_0\to A\to 0$ in mod Λ with each $U_i\in\mathrm{add}_\Lambda U$ for any $i\geq 0$, then we define U-resol.$\mathrm{dim}_\Lambda(A)=\inf\{n|$ there is an exact sequence $0\to U_n\to\cdots\to U_1\to U_0\to A\to 0$ in mod Λ with each $U_i\in\mathrm{add}_\Lambda U$ for any $0\leq i\leq n\}$. We set U-resol.$\mathrm{dim}_\Lambda(A)$ infinity if no such an integer exists. Dually, for a module B in mod Γ^{op}, we may define U-resol.$\mathrm{dim}_\Gamma(B)$ (see [2]).

Lemma 2.7 *Let E be injective in* mod Λ (*resp.* mod Γ^{op}). *Then $l.\mathrm{fd}_\Gamma(^*E)$ (resp. $r.\mathrm{fd}_\Lambda(^*E)$) $\leq n$ if and only if U-resol.$\mathrm{dim}_\Lambda(E)$ (resp. U-resol.$\mathrm{dim}_\Gamma(E)$) $\leq n$.*

Proof. Assume that E is injective in mod Λ and $l.\mathrm{fd}_\Gamma(^*E)\leq n$. Then there is an exact sequence $0\to Q_n\to\cdots\to Q_1\to Q_0\to {}^*E\to 0$ with each Q_i flat (and hence projective) in mod Γ for any $0\leq i\leq n$. By [3] Chapter VI, Proposition 5.3, $\mathrm{Tor}_j^\Gamma(U,{}^*E)\cong\mathrm{Hom}_\Lambda(\mathrm{Ext}_\Gamma^j(U,U),E)=0$ for any $j\geq 1$. Then we easily have an exact sequence:

$$0\to U\otimes_\Gamma Q_n\to\cdots\to U\otimes_\Gamma Q_1\to U\otimes_\Gamma Q_0\to U\otimes_\Gamma {}^*E\to 0.$$

It is clear that $U\otimes_\Gamma Q_i\in\mathrm{add}_\Lambda U$ for any $0\leq i\leq n$. By [9] p.47, $U\otimes_\Gamma {}^*E\cong\mathrm{Hom}_\Lambda(\mathrm{Hom}_\Gamma(U,U),E)\cong E$. Hence we conclude that U-resol.$\mathrm{dim}_\Lambda(E)\leq n$.

Conversely, if U-resol.$\mathrm{dim}_\Lambda(E)\leq n$ then there is an exact sequence $0\to X_n\to\cdots\to X_1\to X_0\to E\to 0$ with each X_i in $\mathrm{add}_\Lambda U$ for any $0\leq i\leq n$. Since $\mathrm{Ext}_\Lambda^j(U,X_i)=0$ for any $j\geq 1$ and $0\leq i\leq n$, $0\to {}^*X_n\to\cdots\to {}^*X_1\to {}^*X_0\to {}^*E\to 0$ is exact with each ${}^*X_i$ $(0\leq i\leq n)$ Γ-projective. Hence we are done. ■

Corollary 2.8 *Let E be injective in* mod Λ (*resp.* mod Γ^{op}). *Then ${}^*E$ is flat in* mod Γ (*resp.* mod Λ^{op}) *if and only if ${}_\Lambda E\in\mathrm{add}_\Lambda U$ (resp. $E_\Gamma\in\mathrm{add}U_\Gamma$).*

From now on, assume that

$$0\to {}_\Lambda U\xrightarrow{f_0}E_0\xrightarrow{f_1}E_1\xrightarrow{f_2}\cdots\xrightarrow{f_i}E_i\xrightarrow{f_{i+1}}\cdots$$

is a minimal injective resolution of ${}_\Lambda U$.

Lemma 2.9 *Suppose* U-dom.dim$({}_\Lambda U) \geq 1$. *Then, for any* $n \geq 2$, U-dom.dim$({}_\Lambda U) \geq n$ *if and only if* $\mathrm{grade}_U M \geq n$ *for any* $M \in \mathrm{mod}\ \Lambda$ *with* $M^* = 0$.

Proof. For any $M \in$mod Λ and $i \geq 1$, we have an exact sequence

$$\mathrm{Hom}_\Lambda(M, E_{i-1}) \to \mathrm{Hom}_\Lambda(M, \mathrm{Im} f_i) \to \mathrm{Ext}^i_\Lambda(M, U) \to 0 \qquad (\dagger)$$

Suppose U-dom.dim$({}_\Lambda U) \geq n$. Then E_i is cogenerated by ${}_\Lambda U$ for any $0 \leq i \leq n-1$. So, for a given $M \in$mod Λ with $M^* = 0$ we have that $\mathrm{Hom}_\Lambda(M, E_i) = 0$ and $\mathrm{Hom}_\Lambda(M, \mathrm{Im} f_i) = 0$ for any $0 \leq i \leq n-1$. Then by the exactness of $(\dagger)$, $\mathrm{Ext}^i_\Lambda(M, U) = 0$ for any $1 \leq i \leq n-1$, and so $\mathrm{grade}_U M \geq n$.

Now we prove the converse, that is, we will prove that $E_i \in \mathrm{add}_\Lambda U$ for any $0 \leq i \leq n-1$.

First, $E_0 \in \mathrm{add}_\Lambda U$ by assumption. We next prove $E_1 \in \mathrm{add}_\Lambda U$. For any $0 \neq x \in \mathrm{Im} f_1$, we claim that $M^* = \mathrm{Hom}_\Lambda(M, U) \neq 0$, where $M = \Lambda x$. Otherwise, we have $\mathrm{Ext}^i_\Lambda(M, U) = 0$ for any $0 \leq i \leq n-1$ by assumption. Since $E_0 \in \mathrm{add}_\Lambda U$, $\mathrm{Hom}_\Lambda(M, E_0) = 0$. So from the exactness of $(\dagger)$ we know that $\mathrm{Hom}_\Lambda(M, \mathrm{Im} f_1) = 0$, which is a contradiction. Then we conclude that $\mathrm{Im} f_1$, and hence E_1, is cogenerated by ${}_\Lambda U$. Notice that E_1 is finitely cogenerated, so $E_1 \in \mathrm{add}_\Lambda U$. Finally, suppose that $n \geq 3$ and $E_i \in \mathrm{add}_\Lambda U$ for any $0 \leq i \leq n-2$. Then by using a similar argument to that above we have $E_{n-1} \in \mathrm{add}_\Lambda U$. The proof is finished. ■

Dually, we have the following

Lemma 2.9′ *Suppose* U-dom.dim$(U_\Gamma) \geq 1$. *Then, for any* $n \geq 2$, U-dom.dim$(U_\Gamma) \geq n$ *if and only if* $\mathrm{grade}_U N \geq n$ *for any* $N \in \mathrm{mod}\ \Gamma^{op}$ *with* $N^* = 0$.

We now are in a position to prove the main result in this paper.

Proof of Theorem 1.3. We only need to prove U-dom.dim$({}_\Lambda U) \leq U$-dom.dim(U_Γ). Without loss of generality, suppose U-dom.dim$({}_\Lambda U) = n$.

The case $n = 1$ follows from Corollaries 2.5 and 2.8. Let $n \geq 2$. Notice that U-dom.dim$({}_\Lambda U) \geq 1$ and U-dom.dim$(U_\Gamma) \geq 1$. By Lemma 2.9′ it suffices to show that $\mathrm{grade}_U N \geq n$ for any $N \in \mathrm{mod}\ \Gamma^{op}$ with $N^* = 0$. By Lemmas 2.3 and 2.7, for any $i \geq 1$, $\mathrm{Hom}_\Lambda(\mathrm{Ext}^i_\Gamma(N, U), E_0) \cong \mathrm{Tor}^\Gamma_i(N, {}^*E_0) = 0$, so $[\mathrm{Ext}^i_\Gamma(N, U)]^* = 0$. Then by assumption and Lemma 2.9, $\mathrm{grade}_U \mathrm{Ext}^i_\Gamma(N, U) \geq n$ for any $i \geq 1$. It follows from Lemma 2.6 that $\mathrm{grade}_U N \geq n$. ■

3. Some applications

As applications to the results in above section, we give in this section some characterizations of $(-)^{**}$ preserving monomorphisms and being left exact respectively.

Assume that

$$0 \to U_\Gamma \xrightarrow{f'_0} E'_0 \xrightarrow{f'_1} E'_1 \xrightarrow{f'_2} \cdots \xrightarrow{f'_i} E'_i \xrightarrow{f'_{i+1}} \cdots$$

is a minimal injective resolution of U_Γ. We first have the following

Proposition 3.1 *The following statements are equivalent for any positive integer k.*

(1) U-dom.dim$({}_\Lambda U) \geq k$.

(2) $0 \to ({}_\Lambda U)^{**} \xrightarrow{f_0^{**}} E_0^{**} \xrightarrow{f_1^{**}} E_1^{**} \cdots \xrightarrow{f_{k-1}^{**}} E_{k-1}^{**}$ *is exact.*

$(1)^{op}$ U-dom.dim$(U_\Gamma) \geq k$.

$(2)^{op}$ $0 \to (U_\Gamma)^{**} \xrightarrow{(f'_0)^{**}} (E'_0)^{**} \xrightarrow{(f'_1)^{**}} (E'_1)^{**} \cdots \xrightarrow{(f'_{k-1})^{**}} (E'_{k-1})^{**}$ *is exact.*

Proof. By Theorem 1.3 we have (1) $\Leftrightarrow$ $(1)^{op}$. By symmetry, we only need to prove (1) $\Leftrightarrow$ (2).

If U-dom.dim$({}_\Lambda U) \geq k$, then E_i is in add${}_\Lambda U$ for any $1 \leq i \leq k-1$. Notice that ${}_\Lambda U$ and each E_i $(0 \leq i \leq k-1)$ are U-reflexive and hence we have that $0 \to ({}_\Lambda U)^{**} \xrightarrow{f_0^{**}} E_0^{**} \xrightarrow{f_1^{**}} E_1^{**} \cdots \xrightarrow{f_{k-1}^{**}} E_{k-1}^{**}$ is exact. Assume that (2) holds. We proceed by induction on k. By assumption we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & {}_\Lambda U \arrow[r, "f_0"] \arrow[d, "\sigma_U"] & E_0 \arrow[r, "f_1"] \arrow[d, "\sigma_{E_0}"] & E_1 \arrow[r, "f_2"] \arrow[d, "\sigma_{E_1}"] & \cdots \arrow[r, "f_{k-1}"] & E_{k-1} \arrow[d, "\sigma_{E_{k-1}}"] \\
0 \arrow[r] & ({}_\Lambda U)^{**} \arrow[r, "f_0^{**}"] & E_0^{**} \arrow[r, "f_1^{**}"] & E_1^{**} \arrow[r, "f_2^{**}"] & \cdots \arrow[r, "f_{k-1}^{**}"] & E_{k-1}^{**}
\end{tikzcd}$$

Since σ_U is an isomorphism, $\sigma_{E_0} f_0 = f_0^{**}\sigma_U$ is a monomorphism. But f_0 is essential, so σ_{E_0} is monic, that is, E_0 is U-torsionless and E_0 is cogenerated by ${}_\Lambda U$. Moreover, E_0 is finitely cogenerated, so we have that $E_0 \in$add${}_\Lambda U$ (and hence σ_{E_0} is an isomorphism). The case $k = 1$ is proved. Now suppose that $k \geq 2$ and $E_i \in$add${}_\Lambda U$ (and then σ_{E_i} is an isomorphism) for any $0 \leq i \leq k-2$. Put $A_0 = {}_\Lambda U$, $B_0 = ({}_\Lambda U)^{**}$, $g_0 = f_0$, $g'_0 = f_0^{**}$ and $h_0 = \sigma_U$. Then, for any $0 \leq i \leq k-2$, we get the following commutative diagrams with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A_i \arrow[r, "g_i"] \arrow[d, "h_i"] & E_i \arrow[r] \arrow[d, "\sigma_{E_i}"] & A_{i+1} \arrow[r] \arrow[d, "h_{i+1}"] & 0 \\
0 \arrow[r] & B_i \arrow[r, "g'_i"] & E_i^{**} \arrow[r] & B_{i+1} \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & A_{i+1} \arrow[r, "g_{i+1}"] \arrow[d, "h_{i+1}"] & E_{i+1} \arrow[d, "\sigma_{E_{i+1}}"] \\
0 \arrow[r] & B_{i+1} \arrow[r, "g'_{i+1}"] & E_{i+1}^{**}
\end{tikzcd}$$

where $A_i = \mathrm{Im} f_i$ and $A_{i+1} = \mathrm{Im} f_{i+1}$, $B_i = \mathrm{Im} f_i^{**}$ and $B_{i+1} = \mathrm{Im} f_{i+1}^{**}$, g_i and g_{i+1} are essential monomorphisms, h_i and h_{i+1} are induced homomorphisms. We may get inductively that each h_j is an isomorphism for any $0 \leq j \leq k-1$. Because $\sigma_{E_{k-1}} g_{k-1} = g'_{k-1} h_{k-1}$ is a monomorphism, by using a similar argument to that above we have $E_{k-1} \in \mathrm{add}_\Lambda U$. Hence we conclude that U-dom.dim$({}_\Lambda U) \geq k$. ■

Proposition 3.2 *The following statements are equivalent.*

(1) U-dom.dim$({}_\Lambda U) \geq 1$.

(2) $(-)^{**} : \mathrm{mod}\ \Lambda \to \mathrm{mod}\ \Lambda$ *preserves monomorphisms.*

(3) $0 \to ({}_\Lambda U)^{**} \xrightarrow{f_0^{**}} E_0^{**}$ *is exact.*

$(1)^{op}$ U-dom.dim$(U_\Gamma) \geq 1$.

$(2)^{op}$ $(-)^{**} : \mathrm{mod}\ \Gamma^{op} \to \mathrm{mod}\ \Gamma^{op}$ *preserves monomorphisms.*

$(3)^{op}$ $0 \to (U_\Gamma)^{**} \xrightarrow{(f'_0)^{**}} (E'_0)^{**}$ *is exact.*

Proof. By Theorem 1.3 we have $(1) \Leftrightarrow (1)^{op}$. By symmetry, we only need to prove that the conditions of (1), (2) and (3) are equivalent.

$(1) \Rightarrow (2)$ If U-dom.dim$({}_\Lambda U) \geq 1$ then $t(X) = \mathrm{Ker}\sigma_X$ for any $X \in \mathrm{mod}\ \Lambda$ by Corollary 2.8 and Proposition 2.4. So $(-)^{**}$ preserves monomorphisms by Proposition 2.2.

$(2) \Rightarrow (3)$ is trivial and $(3) \Rightarrow (1)$ follows from Proposition 3.1. ■

Remark. Proposition 3.2 develops [5] Theorem 1.

Proposition 3.3 *The following statements are equivalent.*

(1) U-dom.dim$({}_\Lambda U) \geq 2$.

(2) $(-)^{**} : \mathrm{mod}\ \Lambda \to \mathrm{mod}\ \Lambda$ *is left exact.*

(3) $0 \to ({}_\Lambda U)^{**} \xrightarrow{f_0^{**}} E_0^{**} \xrightarrow{f_1^{**}} E_1^{**}$ *is exact.*

(4) $(-)^{**} : \mathrm{mod}\ \Lambda \to \mathrm{mod}\ \Lambda$ *preserves monomorphisms and* $\mathrm{Ext}_\Gamma^1(\mathrm{Ext}_\Lambda^1(X, U), U) = 0$ *for any* $X \in \mathrm{mod}\ \Lambda$.

$(1)^{op}$ U-dom.dim$(U_\Gamma) \geq 2$.

$(2)^{op}$ $(-)^{**} : \mathrm{mod}\ \Gamma^{op} \to \mathrm{mod}\ \Gamma^{op}$ *is left exact.*

$(3)^{op}$ $0 \to (U_\Gamma)^{**} \xrightarrow{(f'_0)^{**}} (E'_0)^{**} \xrightarrow{(f'_1)^{**}} (E'_1)^{**}$ *is exact.*

$(4)^{op}$ $(-)^{**} : \mathrm{mod}\ \Gamma^{op} \to \mathrm{mod}\ \Gamma^{op}$ *preserves monomorphisms and* $\mathrm{Ext}_\Lambda^1(\mathrm{Ext}_\Gamma^1(Y, U), U) = 0$ *for any* $Y \in \mathrm{mod}\ \Gamma^{op}$.

Proof. By Theorem 1.3 we have $(1) \Leftrightarrow (1)^{op}$ and by Proposition 3.1 we have $(1) \Leftrightarrow (3)$. So, by symmetry we only need to prove that $(1) \Leftrightarrow (2)$ and $(1) \Rightarrow (4) \Rightarrow (1)^{op}$.

$(1) \Leftrightarrow (2)$ Assume that $(-)^{**} : \mathrm{mod}\ \Lambda \to \mathrm{mod}\ \Lambda$ is left exact. Then, by Proposition 3.2, we have that U-dom.dim$({}_\Lambda U) \geq 1$ and $E_0 \in \mathrm{add}_\Lambda U$.

Let K =Im$(E_0 \to E_1)$ and $v : K \to E_1$ be the essential monomorphism. By assumption and the exactness of the sequences $0 \to U \to E_0 \to K \to 0$ and $0 \to K \xrightarrow{v} E_1$, we have the following exact commutative diagrams:

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r] \arrow[d, "\sigma_U"] & E_0 \arrow[r] \arrow[d, "\sigma_{E_0}"] & K \arrow[r] \arrow[d, "\sigma_K"] & 0 \\
0 \arrow[r] & U^{**} \arrow[r] & E_0^{**} \arrow[r] & K^{**} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "v"] \arrow[d, "\sigma_K"] & E_1 \arrow[d, "\sigma_{E_1}"] \\
0 \arrow[r] & K^{**} \arrow[r, "v^{**}"] & E_1^{**}
\end{tikzcd}$$

where σ_U and σ_{E_0} are isomorphisms. By applying the snake lemma to the first diagram we have that σ_K is monic. Then we know from the second diagram that $\sigma_{E_1} v = v^{**}\sigma_K$ is a monomorphism. However, v is essential, so σ_{E_1} is monic, that is, E_1 is U-torsionless and E_1 is cogenerated by ${}_\Lambda U$. Moreover, E_1 is finitely cogenerated, so we conclude that $E_1 \in$add${}_\Lambda U$.

Conversely, assume that U-dom.dim$({}_\Lambda U) \geq 2$ and $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is an exact sequence in mod Λ. By Proposition 3.2, α^{**} is monic. By assumption, Corollary 2.8 and Lemma 2.3 we have $\mathrm{Hom}_\Gamma(\mathrm{Ext}^1_\Lambda(C, U), E_0) = 0$. Since Coker$\alpha^*$ is isomorphic to a submodule of $\mathrm{Ext}^1_\Lambda(C, U)$, $\mathrm{Hom}_\Gamma(\mathrm{Coker}\alpha^*, E_0) = 0$ and $\mathrm{Hom}_\Gamma(\mathrm{Coker}\alpha^*, U) = 0$. Then, by Theorem 1.3 and Lemma 2.9$'$, grade$_U$Coker$\alpha^* \geq 2$. It follows easily that $0 \to A^{**} \xrightarrow{\alpha^{**}} B^{**} \xrightarrow{\beta^{**}} C^{**}$ is exact.

(1) $\Rightarrow$ (4) Suppose U-dom.dim$({}_\Lambda U) \geq 2$. By Proposition 3.2, $(-)^{**}$: mod $\Lambda \to$ mod Λ preserves monomorphisms. On the other hand, we have that U-dom.dim$(U_\Gamma) \geq 2$ by Theorem 1.3. It follows from Corollary 2.8 and Lemma 2.3 that $\mathrm{Hom}_\Gamma(\mathrm{Ext}^1_\Lambda(X, U), E_0^{'}) = 0$ for any $X \in$mod Λ. So $[\mathrm{Ext}^1_\Lambda(X, U)]^* = 0$ and hence $\mathrm{Ext}^1_\Gamma(\mathrm{Ext}^1_\Lambda(X, U), U) = 0$ by Lemma 2.9$'$.

(4) $\Rightarrow$ (1)^{op} Suppose that (4) holds. Then U-dom.dim$(U_\Gamma) \geq 1$ by Proposition 3.2.

Let A be in mod Λ and B any submodule of $\mathrm{Ext}^1_\Lambda(A, U)$ in mod Γ^{op}. Since U-dom.dim(U_Γ) ≥ 1, $\mathrm{Hom}_\Gamma(\mathrm{Ext}^1_\Lambda(A, U), E_0^{'}) = 0$ by Corollary 2.8 and Lemma 2.3. So $\mathrm{Hom}_\Gamma(B, E_0^{'}) = 0$ and hence $\mathrm{Hom}_\Gamma(B, E_0^{'}/U_\Gamma) \cong \mathrm{Ext}^1_\Gamma(B, U_\Gamma)$. On the other hand, $\mathrm{Hom}_\Gamma(B, E_0^{'}) = 0$ implies $B^* = 0$. Then by [6] Lemma 2.1 we have that $B \cong\mathrm{Ext}^1_\Lambda(\mathrm{Tr}_U B, U)$ with $\mathrm{Tr}_U B$ in mod Λ. By (4), $\mathrm{Hom}_\Gamma(B, E_0^{'}/U) \cong \mathrm{Ext}^1_\Gamma(B, U) \cong \mathrm{Ext}^1_\Gamma(\mathrm{Ext}^1_\Lambda(\mathrm{Tr}_U B, U), U) = 0$. Then by using a similar argument to that in the proof $(2)^{op} \Rightarrow (1)^{op}$ in Proposition 2.2, we have that

$\mathrm{Hom}_\Gamma(\mathrm{Ext}^1_\Lambda(A,U),E_1^{'})=0$ (note: $E_1^{'}$ is the injective envelope of $E_0^{'}/U$). Thus $E_1^{'}\in\mathrm{add}U_\Gamma$ by Lemma 2.3 and Corollary 2.8, and therefore U-dom.dim$(U_\Gamma)\geq 2$. ■

Remark. Proposition 3.3 develops [5] Theorem 2.

Finally we give some equivalent characterizations of U-resol.dim$_\Lambda(E_0)\leq 1$ as follows.

Proposition 3.4 *The following statements are equivalent.*

(1) U-resol.dim$_\Lambda(E_0)\leq 1$.

(2) σ_X *is an essential monomorphism for any* U*-torsionless module* X *in* mod Λ.

(3) f^{**} *is a monomorphism for any monomorphism* $f:X\to Y$ *in* mod Λ *with* Y *U-torsionless.*

(4) grade$_U\mathrm{Ext}^1_\Lambda(X,U)\geq 1$ (*that is,* $[\mathrm{Ext}^1_\Lambda(X,U)]^*=0$) *for any* X *in* mod Λ.

Proof. (1) $\Rightarrow$ (2) Assume that X is U-torsionless in mod Λ. Then $\mathrm{Coker}\sigma_X\cong\mathrm{Ext}^2_\Gamma(\mathrm{Tr}_UX,U)$ by [6] Lemma 2.1. By Lemmas 2.7 and 2.3 we have $\mathrm{Hom}_\Lambda(\mathrm{Coker}\sigma_X,E_0)=\mathrm{Hom}_\Lambda(\mathrm{Ext}^2_\Gamma(\mathrm{Tr}_UX,U),E_0)=0$. Then $\mathrm{Hom}_\Lambda(A,{}_\Lambda U)=0$ for any submodule A of $\mathrm{Coker}\sigma_X$, which implies that any non-zero submodule of $\mathrm{Coker}\sigma_X$ is not U-torsionless.

Let B be a submodule of X^{**} with $X\bigcap B=0$. Then $B\cong B/X\bigcap B\cong(X+B)/X$ is isomorphic to a submodule of $\mathrm{Coker}\sigma_X$. On the other hand, B is clearly U-torsionless. So $B=0$ and hence σ_X is essential.

(2) $\Rightarrow$ (3) Let $f:X\to Y$ be monic in mod Λ with Y U-torsionless. Then $f^{**}\sigma_X=\sigma_Yf$ is monic. By (2), σ_X is an essential monomorphism, so f^{**} is monic.

(3) $\Rightarrow$ (4) Let X be in mod Λ and $0\to Y\xrightarrow{g}P\to X\to 0$ an exact sequence in mod Λ with P projective. It is easy to see that $[\mathrm{Ext}^1_\Lambda(X,U)]^*\cong\mathrm{Ker}g^{**}$. On the other hand, g^{**} is monic by (3). So $\mathrm{Ker}g^{**}=0$ and $[\mathrm{Ext}^1_\Lambda(X,U)]^*=0$.

(4) $\Rightarrow$ (1) Let M be in mod Γ^{op} and $\cdots\to P_1\to P_0\to M\to 0$ a projective resolution of M in mod Γ^{op}. Put $N=\mathrm{Coker}(P_2\to P_1)$. By [6] Lemma 2.1, $\mathrm{Ext}^2_\Gamma(M,U)\cong\mathrm{Ext}^1_\Gamma(N,U)\cong\mathrm{Ker}\sigma_{\mathrm{Tr}_UN}$. On the other hand, since N is U-torsionless, $\mathrm{Ext}^1_\Lambda(\mathrm{Tr}_UN,U)\cong\mathrm{Ker}\sigma_N=0$.

Let X be any finitely generated submodule of $\mathrm{Ext}^2_\Gamma(M,U)$ and $f_1:X\to\mathrm{Ext}^2_\Gamma(M,U)(\cong\mathrm{Ker}\sigma_{\mathrm{Tr}_UN})$ the inclusion, and let f be the composition: $X\xrightarrow{f_1}\mathrm{Ext}^2_\Gamma(M,U)\xrightarrow{g}\mathrm{Tr}_UN$, where g is a monomorphism. By using the same argument as that in the proof of $(2)^{op}\Rightarrow(1)^{op}$ in Proposition 2.2, we get that $f^*=0$. Hence, by applying $\mathrm{Hom}_\Lambda(-,U)$ to the exact sequence $0\to X\xrightarrow{f}\mathrm{Tr}_UN\to\mathrm{Coker}f\to 0$, we have $X^*\cong\mathrm{Ext}^1_\Lambda(\mathrm{Coker}f,U)$. Then $X^{**}\cong[\mathrm{Ext}^1_\Lambda(\mathrm{Coker}f,U)]^*=0$ by (4), which implies that $X^*=0$ since X^* is a direct summand of $X^{***}(=0)$ by [1] Proposition 20.24. Also by using the same argument as that in the proof of $(2)^{op}\Rightarrow(1)^{op}$ in Proposition 2.2, we get that $\mathrm{Hom}_\Lambda(\mathrm{Ext}^2_\Gamma(M,U),E_0)=0$. It follows from

Lemma 2.3 that $l.\mathrm{fd}_\Gamma({}^*E_0) \leq 1$. Therefore U-resol.dim$_\Lambda(E_0) \leq 1$ by Lemma 2.7. ■

Remark. By Theorem 1.3, we have that $E_0 \in \mathrm{add}_\Lambda U$ if and only if $E_0' \in \mathrm{add} U_\Gamma$, that is, U-resol.dim$_\Lambda(E_0) = 0$ if and only if U-resol.dim$_\Gamma(E_0') = 0$. However, in general, we don't have the fact that U-resol.dim$_\Lambda(E_0) \leq 1$ if and only if U-resol.dim$_\Gamma(E_0') \leq 1$ even when ${}_\Lambda U_\Gamma = {}_\Lambda\Lambda_\Lambda$. We use I_0 and I_0' to denote the injective envelope of ${}_\Lambda\Lambda$ and Λ_Λ, respectively. Consider the following example. Let K be a field and Δ the quiver:

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \arrow[r, "\gamma"] & 3 \end{tikzcd}$$

(1) If $\Lambda = K\Delta/(\alpha\beta\alpha)$. Then $l.\mathrm{fd}_\Lambda(I_0) = 1$ and $r.\mathrm{fd}_\Lambda(I_0') \geq 2$. (2) If $\Lambda = K\Delta/(\gamma\alpha, \beta\alpha)$. Then $l.\mathrm{fd}_\Lambda(I_0) = 2$ and $r.\mathrm{fd}_\Lambda(I_0') = 1$.

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